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ADDENDUM

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Abstract. We develop the idea presented in a previous paper by including the case of non-identical coupling terms and show that, under some conditions, separation of the equations is also possible.

For the sake of clarity we shall restrict ourselves to the case of two coupled differential equations of the form

$$(P+f_0)y_0=By_1 \qquad (P+f_1)y_1=Cy_0 \tag{1}$$

in which, as usual, $B(r)$, $C(r)$, $f_0(r)$ and $f_1(r)$ are assumed to be continuous and differentiable, and $B(r)\neq C(r)$.

In the light of the results presented in I (Cao xuan Chuan 1981), it is clear that a complete separation of system (1) is excluded; however, the equations may be decoupled and solved separately by use of the following extended theorem.

Theorem. The equations in (1) may always be decoupled if and only if the quantities $B/(f_1-f_0)$ and $C/(f_1-f_0)$ are simultaneously independent of r.

Proof. We shall use the procedure followed previously by introducing the transformation T defined by

$$T=\begin{pmatrix} 1-a & 1+a \\ -(1+a) & 1-a \end{pmatrix}$$

in which we already know that a is a quantity which must be independent of r and which may be determined from the diagonalisation requirement of the matrix equation

$$T^{-1}\mathscr{P}TT^{-1}Y=T^{-1}(\mathscr{F}+\mathscr{D})TT^{-1}Y \tag{2}$$

where the matrices $\mathscr{P}$, $\mathscr{F}$, $\mathscr{D}$ and Y have already been defined in I. A complete separation of system (1) in the sense of equation (7) of I in the present case is not possible because the quantity a must then satisfy simultaneously the two equations ($\Delta f=f_1-f_0$)

$$(B-C-\Delta f)a^2-2(B+C)a+(B-C+\Delta f)=0 \tag{3}$$

$$(C-B-\Delta f)a^2-2(B+C)a+(C-B+\Delta f)=0. \tag{4}$$

Each of these equations admits two roots: $a'_\pm$ for (3) and $a''_\pm$ for (4), with $a'_\pm\neq a''_\pm$ if

$B \neq C$. It may be verified, on the other hand, that $a'_\pm$ and $a''_\pm$ will be independent of r if and only if the quantities $B/(f_1-f_0)$ and $C/(f_1-f_0)$ are.

Consider for example the case $a = a'_+ = a'$ where system (1) is decoupled in the form

$$[P+\tfrac{1}{2}(f_0+f_1)+P']W'_0 = 0 \qquad [P+\tfrac{1}{2}(f_0+f_1)+P']W'_1 = N'W_0 \tag{5}$$

where

$$P' = -\frac{1}{2}(B+C)\frac{1-a'^2}{1+a'^2} - \Delta f \frac{a'}{1+a'^2}$$

$$N' = \frac{B-C}{2} - \frac{1}{2}\Delta f \frac{1-a'^2}{1+a'^2} - (B+C)\frac{a'}{1+a'^2} \tag{6}$$

$$a' = \left(\frac{B-C}{\Delta f} - 1\right)^{-1}\left\{\frac{B+C}{\Delta f} + \left[\left(\frac{B+C}{\Delta f}\right)^2 - \left(\frac{B-C}{\Delta f}\right)^2 + 1\right]^{1/2}\right\}$$

$$W' = \begin{pmatrix} W'_0 \\ W'_1 \end{pmatrix} \qquad W' = T(a')Y. \tag{7}$$

W'_0 can now be solved separately first; then, replacing it in the second equation of (5), W'_1 may also be determined. The couple y_0, y_1 which are solutions of (1) are therefore recovered by (7).

As a second transformation is possible with $a = a''_+ = a''$

$$a'' = -\left(\frac{B+C}{\Delta f} + 1\right)^{-1}\left\{\frac{B+C}{\Delta f} + \left[\left(\frac{B+C}{\Delta f}\right)^2 - \left(\frac{C-B}{\Delta f}\right)^2 + 1\right]^{1/2}\right\} \tag{8}$$

we obtain a second set of equations

$$[P+\tfrac{1}{2}(f_0+f_1)-P'']W''_0 = N''W''_1 \qquad [P+\tfrac{1}{2}(f_0+f_1)+P'']W''_1 = 0 \tag{9}$$

in which P'' and N'' are obtained from (6) when B is replaced by C and vice versa. Therefore

$$W'' = T(a'')Y. \tag{10}$$

The same reasoning applies to the quantities a'_- and a''_-.

Remarks. It is noteworthy to point out the following particular cases:

(i) $f_0 \neq f_1, B = C$;
(ii) $f_0 = f_1, B \neq C$;
(iii) $f_1 - f_0 \equiv B - C$.

The first case is the only one for which $a = a' = a''$ and corresponds to the case of identical coupling terms discussed in I. It may be verified in fact that all the results obtained in I can be recovered here if we put $a' = a''$ in (7) and (10), confirming therefore the consistency of the present method.

The second and third cases lead to some symmetry in the solution and correspond to the cases $a' = -a''$ and $a' = 1/a''$ respectively.

Reference

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